A Balancing Domain Decomposition Method for Magnetostatic Problems with a Multigrid Strategy

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A balancing domain decomposition (BDD) method is considered as a preconditioner of the iterative domain decomposition method (DDM) for magnetostatic problems. The BDD method enables us to keep convergence properties of the iterative DDM even if the number of subdomains increases. However, in case of magnetostatic problems, the dimension of the coarse problem required in the BDD procedure depends on the number of nodal points of the discretization based on the finite element method. This fact causes that computational costs increase as computational models become larger. Therefore, to reduce the computational costs, a kind of multigrid strategy is introduced into the BDD procedure.

***Index Terms*—balancing domain decomposition method, finite element analysis, magnetostatics, multigrid methods**

I. Introduction

WE HAVE introduced an iterative domain decomposition method (DDM) to solve large scale computational models derived from electromagnetic field problems; see, for example, Kanayama, et al. [3] and Tagami [5]. In Tagami [5], we formulate the iterative DDM based on a mixed formulation of magnetostatic problems introduced in Kikuchi [1] and [2], which enables us to prove unique solvability of the problems and convergency of the approximate solution.

In this paper, we introduce a balancing domain decomposition (BDD) method with a kind of multigrid strategy, which is regarded as a preconditioner of the reduced iterative DDM proposed in Tagami [5]. The BDD method is originally proposed in Mandel [4], where the linear system is positive symmetric. Although the linear system in our case is indefinite, the BDD method enables us to keep the number of iterations of the iterative DDM even if the number of subdomains increases.

However, in case of magnetostatic problems, the dimension of the coarse problem required in the BDD procedure depends on the number of nodal points of the discretization based on the finite element method. This fact causes that computational costs increase as computational models become larger. Therefore, to reduce the computational costs, a kind of multigrid strategy is introduced into the BDD procedure.

II. Iterative DDM of Magnetostatic Problems

Let Ω be a polyhedoral domain with its boundary Γ and the outward unit normal $\boldsymbol{n}$. Let $\boldsymbol{u}$ denote the magnetic vector potential, $\boldsymbol{f}$ an excitation current density, and ν the magnetic reluctivity. Then, the magnetostatic equation with the magnetic vector potential and the Coulomb gauge condition is formulated as follows:

$$\begin{cases} \mathrm{rot}(\nu\,\mathrm{rot}\,\boldsymbol{u}) = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma; \end{cases} \tag{1}$$

for example, see Kikuchi [1].

As usual, let $L^2(\Omega)$ be the space of real functions defined in Ω and 2nd power integlable in Ω; let $H^1(\Omega)$ be the space of functions in $L^2(\Omega)$ with derivatives up to the 1st order. Then, set functional spaces V and Q by

$$\begin{aligned} V &:= \big\{\boldsymbol{v}\in\big(L^2(\Omega)\big)^3; \\ &\qquad \mathrm{rot}\,\boldsymbol{v}\in\big(L^2(\Omega)\big)^3,\ \boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0} \text{ on } \Gamma\big\}, \\ Q &:= \big\{q\in H^1(\Omega); q=0 \text{ on } \Gamma\big\}, \end{aligned} \tag{2}$$

respectively.

Now, following Kikuchi [1], a mixed weak formulation of magnetostatic problems with the Lagrange multiplier p is formulated as follows: given $\boldsymbol{f}\in\big(L^2(\Omega)\big)^3$, find $(\boldsymbol{u},p)\in V\times Q$ such that

$$\begin{cases} (\nu\,\mathrm{rot}\,\boldsymbol{u}, \mathrm{rot}\,\boldsymbol{v}) + (\boldsymbol{v}, \mathrm{grad}\,p) = (\boldsymbol{f},\,\boldsymbol{v}), \\ (\boldsymbol{u}, \mathrm{grad}\,q) = 0, \qquad\qquad \forall(\boldsymbol{v},\,q)\in V\times Q, \end{cases} \tag{3}$$

where $(\,.\,,.\,)$ denotes the inner product of $\big(L^2(\Omega)\big)^3$.

As in Kikuchi [2], magnetic vector potential $\boldsymbol{u}$ is approximated by the Nedelec element of the first order and the Lagrange multiplier p is approximated by the conventional $P1$-element. Then, the resultant linear system is obtained as the finite element equation of magnetostatic problems with the mixed formulation:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \tag{4}$$

where $\mathbf{u}$, $\mathbf{p}$, $\mathbf{f}$ correspond to the degrees of freedom (DOF) of the magnetic vector potential, the Lagrange multiplier, and the excitation current density, respectively.

Now, let us introduce the iterative DDM. The domain Ω is assumed to be decomposed into non-overlapping subdomains $\Omega^{(i)}$ satisfying $\Omega^{(i)}\neq\emptyset$, $\overline{\Omega} = \bigcup_i \overline{\Omega^{(i)}}$, $\Omega^{(i)}\cap\Omega^{(j)} = \emptyset\ (i\neq j)$; and let γ_B be the interface defined by $\gamma_B := \bigcup_{i\neq j}\big(\overline{\Omega^{(i)}}\cap\overline{\Omega^{(j)}}\big)$. Then, following Tagami [5], the main part of a reduced iterative DDM for magnetostatic problems is solving the following interface problem;

$$\mathbf{S}\mathbf{u}_B = \mathbf{r}, \tag{5}$$

where $\mathbf{S}$ denotes the Schur complement matrix defined by

$$\mathbf{S} := \sum_i \mathbf{E}^{(i)}\mathbf{S}^{(i)}{\mathbf{E}^{(i)}}^T, \tag{6}$$

$$\mathbf{S}^{(i)} := \mathbf{A}_{II}^{(i)} - \mathbf{A}_{IB}^{(i)}\mathbf{A}_{BB}^{\dagger}{\mathbf{A}_{IB}}^T, \tag{7}$$

$$\mathbf{A}^{(i)} := \begin{pmatrix} \mathbf{A}_{II}^{(i)} & \mathbf{A}_{IB}^{(i)} \\ {\mathbf{A}_{IB}^{(i)}}^T & \mathbf{A}_{BB}^{(i)} \end{pmatrix}, \tag{8}$$

$\mathbf{E}^{(i)}$ denotes the prolongation from DOF on γ_B to DOF in Ω, the superscripts (i) denote the matrices corresponding to $\Omega^{(i)}$, the subscripts I and B denote DOF corresponding to $\Omega^{(i)}$ and γ_B, and $\dagger$ denotes the Moore–Penrose generalized inverse, respectively; for detail, see [4].

In practical implementation, we apply the Conjugate Gradient (CG) method into (5). Once $\mathbf{u}_B$ can be obtained, we can solve each subdomain problem, where imposed $\mathbf{u}_B$ as the boundary condition on γ_B. Therefore, we can reconstruct the solution in the whole domain Ω.

III. A Balancing Domain Decomposition Method

As in Mandel [4], a BDD method is formally applied into the interface problem (5) derived from the iterative DDM for magnetostatic problems with the mixed formulation, which is also regarded as a preconditioner of the iterative DDM in Tagami [5]. That is, the preconditioning matirx $\mathbf{M}$ can be written as follows:

$$\mathbf{M}^{\dagger} := \big((\mathbf{I}-\mathbf{P})\mathbf{T}\mathbf{S}(\mathbf{I}-\mathbf{P}) + \mathbf{P}\big)\mathbf{S}^{\dagger}, \tag{9}$$

where

$$\mathbf{T} := \sum_i \mathbf{E}^{(i)}\mathbf{D}_i{\mathbf{S}^{(i)}}^{\dagger}{\mathbf{D}^{(i)}}^T{\mathbf{E}^{(i)}}^T, \tag{10}$$

$$\mathbf{P} := \text{ a projection matrix on } \mathbf{W}, \tag{11}$$

$$\mathbf{W} := \Big\{\mathbf{v} = \sum_i \mathbf{E}^{(i)}\mathbf{D}^{(i)}\mathbf{u}^{(i)},\ \mathbf{u}^{(i)} \in \mathrm{Range}\big(\mathbf{Z}^{(i)}\big)\Big\}, \tag{12}$$

and $\mathbf{Z}^{(i)}$ denotes an appropriate matrix satisfying

$$\mathrm{Ker}\big(\mathbf{S}^{(i)}\big) \subset \mathrm{Range}\big(\mathbf{Z}^{(i)}\big). \tag{13}$$

As in (13), in the BDD methods, the kernel of a coefficient matrix plays an important role. In case of (5), the kernel consists of gradients of the $P1$-approximate Lagrange multipliers. Therefore, the dimension of the kernel is equal to the number of the nodal points of finite element mesh. This is why the computational cost increases as the scale of computational models becomes lager. In order to settle this difficulty, a kind of multigrid strategy is introduced into the BDD method: set a coarse grid in each subdomain, and "1st order polynomials on the coarse grid" are adopted into solving the coarse grid problems in the BDD procedure. That is, the matrix $\mathbf{Z}^{(i)}$ is replaced into $\widetilde{\mathbf{Z}}^{(i)}$, whose range consists of the gradient of the "$P1$-element" with respect to each subdomain $\Omega^{(i)}$, although the condition (13) does not satisfy precisely. At that time, $\mathbf{W}$ and $\mathbf{P}$ are replaced into

$$\widetilde{\mathbf{W}} := \Big\{\widetilde{\mathbf{v}} = \sum_i \mathbf{E}^{(i)}\mathbf{D}^{(i)}\widetilde{\mathbf{u}}^{(i)},\ \widetilde{\mathbf{u}}^{(i)} \in \mathrm{Range}\big(\widetilde{\mathbf{Z}}^{(i)}\big)\Big\}, \tag{14}$$

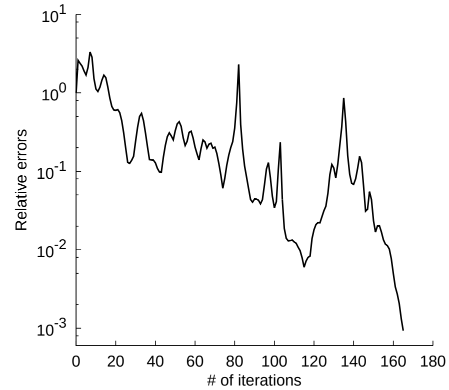

Fig. 1. Convergence history of the CG method for (5).

$$\widetilde{\mathbf{P}} := \text{ a projection matrix on } \widetilde{\mathbf{W}}, \tag{15}$$

respectively. Then, the preconditioner $\mathbf{M}$ is also replaced into $\widetilde{\mathbf{M}}$ defined by

$$\widetilde{\mathbf{M}}^{\dagger} := \big((\mathbf{I}-\widetilde{\mathbf{P}})\mathbf{T}\mathbf{S}(\mathbf{I}-\widetilde{\mathbf{P}}) + \widetilde{\mathbf{P}}\big)\mathbf{S}^{\dagger}, \tag{16}$$

This strategy enable us to reduce computational costs when solving the coarse grid problems, and to keep the efficiency of the BDD method.

IV. Numerical Results

We consider a simple case as numerical examples, where we obtain the exact solution in the circle domain. We use a small scale computational model as the test case: its numbers of DOF is about 10^4. The CG method is stopped when the residual norm becomes less than 10^{-3}.

Fig. 1 shows the convergence history of the CG method for (5). We can confirm to obtain a convergence result by using the proposed BDD method. As far as we know, there are a few results on convergences of BDD methods for magnetostatic or eddy current problems. Therefore, Fig. 1 is considered as the first step of BDD methods for magnetic field problems. We have no space enough to compare with other DDM in case of more larger scale computational models. We show these results at the poster.

V. Conclusions

The Balancing Domain Decomposition (BDD) method with a kind of multigrid methods has been applied to magnetostatic problems. We continue to confirm efficiency of the BDD method, and apply the BDD method into more practical computational models.

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